The Hitchhiker′s Guide to the Hydrodynamics

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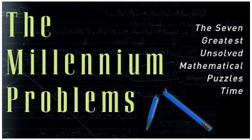
- Y. Hatta, J. Noronha, BX, Phys.Rev. D89 (2014) 051702.
- Y. Hatta, J. Noronha, BX, 1403.7693.
- Y. Hatta, BX, 1405.1984.

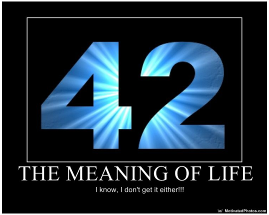
Hydrodynamics

- Non-relativistic hydro is very important to our day-to-day life.
- Relativistic hydrodynamics is widely used in astrophysics and cosmology and study of quark gluon plasma in heavy ion collisions.
- Navier Stokes (1st order hydro-equation) existence and smoothness. Hard!



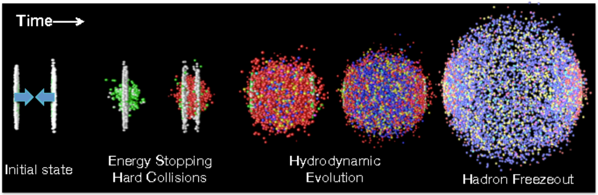
- Study of hydrodynamics heavily rely on numerical methods.

The Hitchhiker's Guide

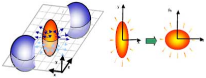


- The Hitchhiker's Guide to the ~~Galaxy~~ Hydrodynamics.
- Miklos Gyulassy: one of the most philosophical and entertaining movie.
- Most importantly, it tells you "Don't Panic".
- Numerical approach sometimes is like a "black box" to non-experts. Like "42".
- "42" is the simple "answer to the Ultimate Question of Life, the Universe, and Everything", calculated by an enormous supercomputer over exactly 7.5 million years.
- It will be nice to have some exact solutions and "Analytic Insights", once for a while.

Relativistic Hydrodynamics



- Few analytical exact solutions to hydrodynamics equations in general.
- Ideal and viscous relativistic hydrodynamics widely used in heavy ion collisions.
- The elliptic flow v_2 is one of the most important signature of the quark gluon plasma created in HIC.
- Our following work will provide analytical insight to the onset of the flow. Please stay tuned. Y. Hatta, BX, 1405.1984.



The Hitchhiker's Guide to the Hydrodynamics

Bo-Wen Xiao

Introduction

Conformal Soliton Solution

Rotating Flow

Bjorken Flow at Second Order

Elliptic flow

Summary

Classification of Relativistic Hydrodynamic equations

- Ideal (inviscid) relativisitic hydrodynamics for pefect fluid

$$\begin{aligned} \nabla_\mu T^{\mu\nu} &= 0 \\ \text{with} \quad T^{\mu\nu} &= \epsilon u^\mu u^\nu + P\Delta^{\mu\nu} \quad \text{and} \quad \Delta^{\mu\nu} = g^{\mu\nu} + u^\mu u^\nu \end{aligned}$$

  - Equation of state relates ϵ and P. Conformal invariance $\Rightarrow T^\mu{}_\mu = 0 \Rightarrow \epsilon = 3P$.
  - Many exact solutions. Biro, Csorgo, Nagy, Csernai, Csanad, Hama, Kodama, Peschanski, Janik, Bialas, Beuf, Saridakis, Liao, Koch, Lin, Oz$\cdots$

- Relativistic Navier-Stokes equation (1st order in ∂u)

$$T^{\mu\nu} = \epsilon u^\mu u^\nu + P\Delta^{\mu\nu} + \Pi^{\mu\nu}, \quad \text{with} \quad \Pi^{\mu\nu} = -2\eta\sigma^{\mu\nu}$$

  - Gubser,09,10 Exact solution.
  - This equation is pathological because it often violates causality and it is unstable.

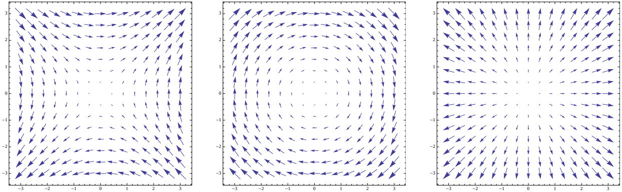
- Second order ($\partial^2 u$) relativistic hydrodynamics $\Leftarrow$ This talk, exact solutions.
  - Causality and stability restored by the Israel-Stewart equation Marrochio,Noronha,Denicol,Luzum,Jeon,Gale (2013)

$$\Pi^{\mu\nu} = -2\eta\sigma^{\mu\nu} - \tau_\pi\left[\Delta^\mu_\alpha\Delta^\nu_\beta D\Pi^{\alpha\beta} + \Pi^{\mu\nu}\theta\right] + \lambda_2\Pi^{\langle\mu}{}_\lambda\Omega^{\nu\rangle\lambda}$$

  - Full Second order equation Denicol, Niemi, Molnar, Rischke (2012).

Three basic flows

Shear flow $\partial_x u_y = \partial_y u_x$ Rotating flow $\partial_x u_y = -\partial_y u_x$ Radial flow



$$\sigma^{\mu\nu} \equiv \nabla^{\langle\mu}u^{\nu\rangle} \equiv \left(\frac{1}{2}(\Delta^{\mu\alpha}\Delta^{\nu\beta} + \Delta^{\mu\beta}\Delta^{\nu\alpha}) - \frac{1}{3}\Delta^{\mu\nu}\Delta^{\alpha\beta}\right)\nabla_\alpha u_\beta\,,$$

$$\Omega^{\mu\nu} \equiv \tfrac{1}{2}\Delta^{\mu\alpha}\Delta^{\nu\beta}(\nabla_\alpha u_\beta - \nabla_\beta u_\alpha)$$

- Represented by $\sigma^{\mu\nu}$ (symmetric), $\Omega^{\mu\nu}$ and $\theta \equiv \nabla_\mu u^\mu = \partial_\mu u^\mu + \Gamma^\mu_{\mu\nu}u^\nu$.
- $\Omega^{\mu\nu}$ is antisymmetric, this is why it can only show up at second order.
- u^μ is the four velocity of the flow. $u^\mu u_\mu = -1$. Static $u_\mu = (-1, 0, 0, 0)$
- $\Delta^{\mu\nu} = g^{\mu\nu} + u^\mu u^\nu$ is the projection operator. $\Delta^{\mu\nu}u_\nu = 0$

The Hitchhiker's Guide to the Hydrodynamics

Bo-Wen Xiao

Introduction

Conformal Soliton Solution

Rotating Flow

Bjorken Flow at Second Order

Elliptic flow

Summary

- Energy momentum conservation $\nabla_\mu T^{\mu\nu} = 0$.

$$T^{\mu\nu} = \epsilon u^\mu u^\nu + P\Delta^{\mu\nu} + \Pi^{\mu\nu}.$$

- Project to $u_\nu \Rightarrow$

$$D\epsilon + (\epsilon + p)\vartheta + \Pi^{\mu\nu}\sigma_{\mu\nu} = 0$$

with comoving derivative $D \equiv u^\mu \nabla_\mu$.

- Project to direction perpendicular ($\perp$) to $u_\nu \Rightarrow$ (use $\Delta^{\mu\nu} = g^{\mu\nu} + u^\mu u^\nu$ since $\Delta^{\mu\nu} u_\nu = 0$)

$$(\epsilon + p)Du^\mu + \Delta^{\mu\alpha}\nabla_\alpha p + \Delta^\mu{}_\nu \nabla_\alpha \Pi^{\alpha\nu} = 0$$

- Landau-Lifshitz frame (momentum density is due to the flow of energy density)

$$u_\mu T^{\mu\nu} = \epsilon u^\nu \quad \Rightarrow \quad u_\mu \Pi^{\mu\nu} = 0.$$

- $\Pi^\mu{}_\mu = 0$ traceless. Also note that

$$u_\nu \nabla_\mu \Pi^{\mu\nu} = -\Pi^{\mu\nu}\nabla_\mu u_\nu = -\Pi^{\mu\nu}\sigma_{\mu\nu} \neq 0$$

Conformal Hydrodynamics

Equations in question: [Baier, Romatschke, Son, Starinets, Stephanov, 07]

$$
\begin{aligned}
\nabla_\mu T^{\mu\nu} &= 0 \quad \text{and} \quad \epsilon = 3p \\
T^{\mu\nu} &= \epsilon u^\mu u^\nu + P\Delta^{\mu\nu} + \Pi^{\mu\nu},
\end{aligned}
$$

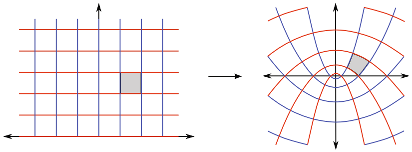
with independent variables in d dimension

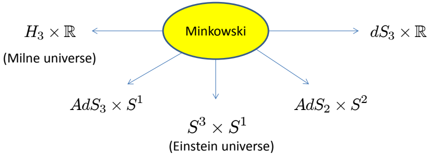
$$
\begin{aligned}
\Pi^{\mu\nu} = -\eta\sigma^{\mu\nu} - \tau_\Pi \left[\langle D\Pi^{\mu\nu} \rangle + \frac{d}{d-1} \Pi^{\mu\nu} (\nabla \cdot u) \right] \\
+ \kappa \left[R^{\langle\mu\nu\rangle} - (d-2) u_\alpha R^{\alpha\langle\mu\nu\rangle\beta} u_\beta \right] \\
+ \frac{\lambda_1}{\eta^2} \Pi^{\langle\mu}{}_\lambda \Pi^{\nu\rangle\lambda} - \frac{\lambda_2}{\eta} \Pi^{\langle\mu}{}_\lambda \Omega^{\nu\rangle\lambda} + \lambda_3 \Omega^{\langle\mu}{}_\lambda \Omega^{\nu\rangle\lambda} .
\end{aligned}
$$


- This equation is conformal invariant, which means it is the same in different metrics which are related by conformal transform.
- Use the conformal symmetry to help us to find exact solutions!

Technique

Conformal Transform (Weyl rescaling)





[Hatta]

- From Minkowski space-time, use conformal transform or coordinate transformation to go to curved spacetimes. $ds^2 = \Lambda^2 \hat{g}_{\mu\nu} d\hat{x}^\mu d\hat{x}^\nu \equiv \Lambda^2 d\hat{s}^2$
- Starting from hydrostatic solutions or simple solution with rotation, find the solutions. The second order equation becomes simple.
- Transform back to Minkowski space-time. $ds^2 = -dt^2 + dx^2 + dy^2 + dz^2$

Conformal Transform

- Minkowski space $\Rightarrow AdS_3 \times S^1$

$$ds^2 = -dt^2 + dz^2 + dx_\perp^2 + x_\perp^2 d\phi^2 = x_\perp^2 [\underbrace{\frac{-dt^2 + dz^2 + dx_\perp^2}{x_\perp^2}}_{AdS_3} + \underbrace{d\phi^2}_{S^1}]$$

- Minkowski space $\Rightarrow AdS_2 \times S^2$

$$ds^2 = -dt^2 + dr^2 + r^2 d\Omega^2 = r^2 (\underbrace{\frac{-dt^2 + dr^2}{r^2}}_{AdS_2} + \underbrace{d\Omega^2}_{S^2}) .$$

- Minkowski space $\Rightarrow dS_3 \times \mathbb{R}$

$$\begin{aligned} d\hat{s}^2 \equiv \frac{ds^2}{\tau^2} &= \frac{-d\tau^2 + dx_\perp^2 + x_\perp^2 d\phi^2}{\tau^2} + d\eta^2 \\ &= \underbrace{-d\varrho^2 + \cosh^2 \varrho (d\Theta^2 + \sin^2 \Theta d\phi^2)}_{dS_3} + \underbrace{d\eta^2}_{\mathbb{R}} \end{aligned}$$

- Change coordinates in Minkowski:

$$ds^2 = -d\tau^2 + \tau^2 d\eta^2 + dx^2 + dy^2 \quad \text{with} \quad \tau \equiv \sqrt{t^2 - z^2}\,, \quad \eta \equiv \tanh^{-1} \frac{z}{t}$$

The Hitchhiker′s Guide to the Hydrodynamics

Bo-Wen Xiao

Introduction

Conformal Soliton Solution

Rotating Flow

Bjorken Flow at Second Order

Elliptic flow

Summary

Anti de Sitter space

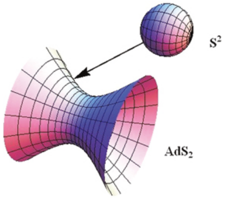
The AdS_2 space is a 2-dim hypersurface in 3-dim:

$$X_0^2 - X_1^2 + X_2^2 = L^2.$$

$$\begin{cases} X_1 = \frac{tL}{r} = L\cosh\rho\cos T \\ X_2 = \frac{(L^2-r^2+t^2)L}{2r} = L\sinh\tilde{\rho} \\ X_3 = \frac{(L^2+r^2-t^2)L}{2r} = L\cosh\tilde{\rho}\sin T \end{cases} \Rightarrow$$

$$\begin{cases} \cosh\tilde{\rho} \equiv \frac{1}{2Lr}\sqrt{(L^2+(r+t)^2)(L^2+(r-t)^2)} \\ \tan T = \frac{L^2+r^2-t^2}{2Lt} \end{cases}$$

$$d\hat{s}^2 = \underbrace{-\cosh^2\tilde{\rho}\,dT^2 + d\tilde{\rho}^2}_{AdS_2} + \underbrace{d\theta^2 + \sin^2\theta d\phi^2}_{S^2}$$



The Hitchhiker′s Guide to the Hydrodynamics

Bo-Wen Xiao

Introduction

Conformal Soliton Solution

Rotating Flow

Bjorken Flow at Second Order

Elliptic flow

Summary

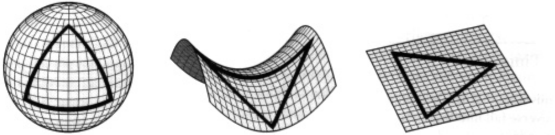
Anti de Sitter(AdS) space is a maximally symmetric, vacuum solution of Einstein's field equation with a negative constant curvature.

The AdS_3 space is a 3-dim hypersurface in 4-dim (Hyperbolic geometry):

$$X_0^2 - X_1^2 - X_2^2 + X_3^2 = L^2.$$

$$\begin{cases} X_1 = \frac{t}{x_\perp} L = L\cosh\rho\cos\tau \\ X_2 = \frac{z}{x_\perp} L = L\sinh\rho\sin\Theta \\ X_3 = \frac{L^2 - r^2 + t^2}{2x_\perp} L = L\sinh\rho\cos\Theta \\ X_4 = \frac{L^2 + r^2 - t^2}{2x_\perp} L = L\cosh\rho\sin\tau \end{cases} \Rightarrow$$

$$d\hat{s}^2 = -\cosh^2\rho\, d\tau^2 + d\rho^2 + \sinh^2\rho d\Theta^2 + d\phi^2$$



Positive Curvature Negative Curvature Flat Curvature

The Hitchhiker′s Guide to the Hydrodynamics

Bo-Wen Xiao

Introduction

Conformal Soliton Solution

Rotating Flow

Bjorken Flow at Second Order

Elliptic flow

Summary

In $AdS_3 \times S^1$

$$\hat{T}^{\mu\nu} = \hat{\epsilon}\,\hat{u}^\mu \hat{u}^\nu + \frac{\hat{\epsilon}}{3}\hat{\Delta}^{\mu\nu} + \hat{\pi}^{\mu\nu} \rightarrow$$

$$\hat{D}\hat{\epsilon} = 0\,, \quad 4\hat{\epsilon}\,\hat{D}\hat{u}^\mu + \hat{\Delta}^{\mu\nu}\hat{\nabla}_\nu \epsilon + 3\hat{\Delta}^\mu_\nu \hat{\nabla}_\alpha \hat{\pi}^{\nu\alpha} = 0\,,$$

with the static flow

$$\hat{u}_\tau = -\cosh\rho\,, \qquad \hat{u}^\rho = \hat{u}^\Theta = \hat{u}^\phi = 0\,,$$

which gives $\hat{\theta} = \hat{\sigma}_{\mu\nu} = \Omega_{\mu\nu} = 0$.

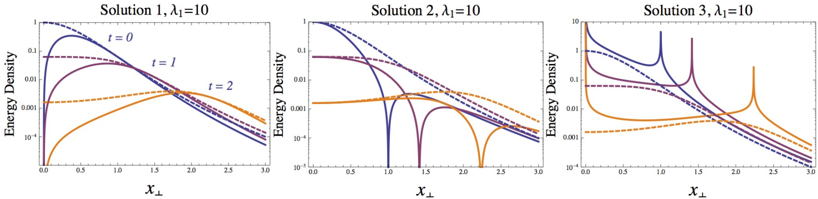
$$\begin{aligned}
\hat{\pi}^{\mu\nu} &= -\frac{\tau_\pi}{\hat{\epsilon}^{1/4}}\hat{\Delta}^\mu_\alpha \hat{\Delta}^\nu_\beta \hat{D}\hat{\pi}^{\alpha\beta} + \frac{\lambda_1}{\hat{\epsilon}}\hat{\pi}^{\langle\mu}{}_\lambda \hat{\pi}^{\nu\rangle\lambda} + \frac{\lambda_2}{\hat{\epsilon}^{1/4}}\hat{\pi}^{\langle\mu}{}_\lambda \hat{\Omega}^{\nu\rangle\lambda} \\
&\quad + \lambda_3 \hat{\epsilon}^{1/2}\hat{\Omega}^{\langle\mu}{}_\lambda \hat{\Omega}^{\nu\rangle\lambda} + \kappa\hat{\epsilon}^{1/2}\left(\hat{\mathcal{R}}^{\langle\mu\nu\rangle} - 2\hat{u}_\alpha \hat{\mathcal{R}}^{\alpha\langle\mu\nu\rangle\beta}\hat{u}_\beta\right)\,, \\
\Rightarrow \hat{\pi}^{\mu\nu} &= \frac{\lambda_1}{\hat{\epsilon}}\hat{\pi}^{\langle\mu}{}_\lambda \hat{\pi}^{\nu\rangle\lambda}
\end{aligned}$$

- The transport coefficients τ_π, κ, λ_i $(i = 1, 2, 3)$ are now dimensionless, and are rescaled by the appropriate power of $\hat{\epsilon}$.

Three solutions in $AdS_3 \times S^1$

Assuming $\hat{\pi}^{\mu\nu}$ is diagonal, we find the solution

$$(\hat{\pi}^{\rho\rho}, \sinh^2\rho\,\hat{\pi}^{\Theta\Theta}, \hat{\pi}^{\phi\phi}) = \frac{\hat{\epsilon}}{\lambda_1} \times \begin{cases} (-1,-1,2)\,, \\ (-1,2,-1)\,, \\ (2,-1,-1)\,. \end{cases}$$



$$\epsilon \propto \begin{cases} \frac{1}{(L^2+(t+r)^2)^2(L^2+(t-r)^2)^2}\left(\frac{4L^2x_\perp^2}{(L^2+(t+r)^2)(L^2+(t-r)^2)}\right)^{\frac{9}{2(\lambda_1-3)}}\,, \\ \frac{1}{(L^2+(t+r)^2)^2(L^2+(t-r)^2)^2}\left(1-\frac{4L^2x_\perp^2}{(L^2+(t+r)^2)(L^2+(t-r)^2)}\right)^{\frac{9}{2(\lambda_1-3)}}\,, \\ \frac{1}{(L^2+(t+r)^2)^2(L^2+(t-r)^2)^2}\left(\frac{4L^2x_\perp^2\left((L^2+(t+r)^2)(L^2+(t-r)^2)-4L^2x_\perp^2\right)}{(L^2+(t+r)^2)^2(L^2+(t-r)^2)^2}\right)^{-\frac{9}{2(\lambda_1+6)}}\,. \end{cases}$$

- $Re^{-1} = \sqrt{\hat{\pi}^{\mu\nu}\hat{\pi}_{\mu\nu}}/\hat{\epsilon} \sim 1/\lambda_1$.
- $Re \sim \lambda_1 \to \infty \Rightarrow$ ideal hydro solution.

Including rotation and vortex

- Use $AdS_3 \times S^1$ metric, (similar solution found in other metric)

$$d\hat{s}^2 = -\cosh^2\rho\, d\tau^2 + d\rho^2 + \sinh^2\rho d\Theta^2 + d\phi^2.$$

- Turn on the rotation to include vortexes

$$\hat{u}_\tau = \frac{-\cosh^2\rho}{\sqrt{\cosh^2\rho - \omega^2}}, \quad \hat{u}_\phi = \frac{\omega}{\sqrt{\cosh^2\rho - \omega^2}}.$$

- When $\omega = 0$, reduces to static solution.
- Ideal hydro solution $\hat{\epsilon} \propto \frac{1}{(\cosh^2\rho - \omega^2)^2}$


Including rotation and vortex

- Turn on the rotation to include vortexes

$$\hat{\pi}^{\mu\nu} = -\frac{\tau_\pi}{\hat{\epsilon}^{1/4}}\hat{\Delta}^\mu_\alpha\hat{\Delta}^\nu_\beta\hat{D}\hat{\pi}^{\alpha\beta} + \frac{\lambda_1}{\hat{\epsilon}}\hat{\pi}^{\langle\mu}{}_\lambda\hat{\pi}^{\nu\rangle\lambda} + \frac{\lambda_2}{\hat{\epsilon}^{1/4}}\hat{\pi}^{\langle\mu}{}_\lambda\hat{\Omega}^{\nu\rangle\lambda} + \lambda_3\sqrt{\hat{\epsilon}}\,\hat{\Omega}^{\langle\mu}{}_\lambda\hat{\Omega}^{\nu\rangle\lambda}$$

- Use $\hat{u}_\mu\hat{\pi}^{\mu\nu} = 0$ and assume

$$\hat{\pi}^{\mu\nu} = \begin{bmatrix} \hat{\pi}^{\tau\tau} & 0 & 0 & \hat{\pi}^{\tau\phi} \\ 0 & \hat{\pi}^{\rho\rho} & 0 & 0 \\ 0 & 0 & \hat{\pi}^{\Theta\Theta} & 0 \\ \hat{\pi}^{\tau\phi} & 0 & 0 & \hat{\pi}^{\phi\phi} \end{bmatrix}.$$

- The solutions are given by

$$(\hat{\pi}^{\rho\rho}, \sinh^2\rho\,\hat{\pi}^{\Theta\Theta}, \hat{\pi}^{\phi\phi}) = \frac{\hat{\epsilon}}{\lambda_1}\left(\alpha, \beta, \frac{\gamma\cosh^2\rho}{\cosh^2\rho - \omega^2}\right),$$

with
$$\alpha = \gamma = -\frac{\beta}{2} = \begin{cases} \frac{1}{2}\left(-1 - \sqrt{1 + 4f/3}\right), \\ \frac{1}{2}\left(-1 + \sqrt{1 + 4f/3}\right), \end{cases}$$

$$\hat{\pi}^{\tau\tau} = \frac{\omega}{\cosh^2\rho}\hat{\pi}^{\tau\phi} = \frac{\omega^2}{\cosh^4\rho}\hat{\pi}^{\phi\phi}, \quad f \equiv \frac{\lambda_1\lambda_3\omega^2\sinh^2\rho}{\sqrt{\hat{\epsilon}}(\cosh^2\rho - \omega^2)^2}.$$

Solving for energy density

The Hitchhiker′s Guide to the Hydrodynamics

Bo-Wen Xiao

Introduction

Conformal Soliton Solution

Rotating Flow

Bjorken Flow at Second Order

Elliptic flow

Summary

Eventually, $\nabla_\mu T^{\mu\nu} = 0 \quad \Rightarrow$

$$\partial_\rho \hat{\epsilon} + \frac{4\cosh\rho\sinh\rho}{\cosh^2\rho - \omega^2}\hat{\epsilon} + 3\left[\partial_\rho \hat{\pi}^{\rho\rho} + \frac{4\cosh\rho\sinh\rho}{\cosh^2\rho - \omega^2}\hat{\pi}^{\rho\rho}\right] + \frac{9(1-\omega^2)\coth\rho}{\cosh^2\rho - \omega^2}\hat{\pi}^{\rho\rho} = 0\,,$$

with

$$\hat{\pi}^{\rho\rho} = \frac{\hat{\epsilon}}{\lambda_1}\alpha\,,$$

$$\text{with} \qquad \alpha = \begin{cases} \frac{1}{2}\left(-1-\sqrt{1+4f/3}\right)\,, \\ \frac{1}{2}\left(-1+\sqrt{1+4f/3}\right)\,, \end{cases}$$

$$f \equiv \frac{\lambda_1\lambda_3\omega^2\sinh^2\rho}{\sqrt{\hat{\epsilon}}(\cosh^2\rho-\omega^2)^2}\,.$$

To solve this, we employ an ansatz

$$\hat{\epsilon} = \frac{A^2\sinh^4\rho}{(\cosh^2\rho-\omega^2)^4}\,, \quad \Rightarrow \quad A = \frac{7\lambda_3\omega^2}{4\left(\frac{4}{21}\lambda_1 - 1\right)}.$$

General solutions are available. Also it reduces to the ideal solution when $\lambda_1 \to \infty$.

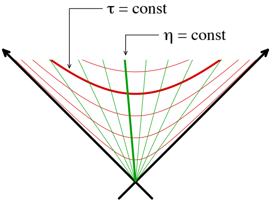
Ideal Bjorken flow

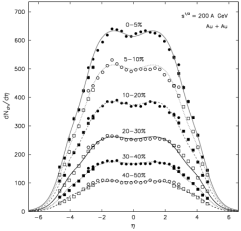
Bjorken flow simply derives from the assumption that flow expands only along z direction and it is independent of rapidity.

$$ds^2 = -d\tau^2 + \tau^2 d\eta^2 + dx^2 + dy^2\,, \qquad u_\tau = -1\,, \qquad u_\eta = u_x = u_y = 0\,.$$

$\tau = \sqrt{t^2 - z^2}$ and rapidity $\eta = \tanh^{-1}\frac{z}{t}$ (simple change of coordinates).

$$\vartheta = \frac{1}{\tau}\,, \qquad \sigma^\eta{}_\eta = \frac{2}{3\tau}\,, \qquad \sigma^x{}_x = \sigma^y{}_y = -\frac{1}{3\tau}\,, \qquad \Omega^{\mu\nu} = 0\,.$$





$$\nabla_\mu T^{\mu\nu} = 0 \quad \Rightarrow \quad u^\mu \nabla_\mu \epsilon + \frac{4}{3}\epsilon\theta = 0 \quad \Rightarrow \quad \epsilon \propto \frac{1}{\tau^{4/3}}\,.$$

The Hitchhiker′s Guide to the Hydrodynamics
Bo-Wen Xiao
Introduction
Conformal Soliton Solution
Rotating Flow
Bjorken Flow at Second Order
Elliptic flow
Summary

Bjorken flow at Second Order

Use the same flow velocity, now solve the second order equation.

$$u_\tau = -1\,, \qquad u_\eta = u_x = u_y = 0\,.$$

$$\vartheta = \frac{1}{\tau}\,, \qquad \sigma^\eta{}_\eta = \frac{2}{3\tau}\,, \qquad \sigma^x{}_x = \sigma^y{}_y = -\frac{1}{3\tau}\,, \qquad \Omega^{\mu\nu} = 0\,.$$

Now the second order hydrodynamic equation becomes

$$\Pi^{\mu\nu} = -2\eta\epsilon^{3/4}\sigma^{\mu\nu} - \frac{\tau_\pi}{\epsilon^{1/4}}\left[\Delta^\mu_\alpha\Delta^\nu_\beta u^\lambda\nabla_\lambda\Pi^{\alpha\beta} + \frac{4}{3}\Pi^{\mu\nu}\vartheta\right] + \frac{\lambda_1}{\epsilon}\Pi^{\langle\mu}_\lambda\hat{\Pi}^{\nu\rangle\lambda},$$

with

$$u^\mu\nabla_\mu\epsilon + \frac{4}{3}\epsilon\vartheta + \Pi^{\mu\nu}\sigma_{\mu\nu} = 0,$$

Perturbative solution [Baier, et al, 07]

$$\epsilon(\tau) \propto \tau^{-\frac{4}{3}}\left[1 - 2\eta\tau^{-\frac{2}{3}} + \left(\frac{3}{2}\eta^2 - \frac{2}{3}\eta\tau_\Pi + \frac{2}{3}\lambda_1\right)\tau^{-\frac{4}{3}} + \cdots\right]$$

[Heller, Janik, Witaszczyk, 13, PRL] It is an asymptotic series with zero radius of convergence. Perturbation may not work.

Bjorken flow at Second Order

We assume that

$$\Pi^{\mu\nu} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & \Pi^{\eta\eta} & 0 & 0 \\ 0 & 0 & \Pi^{xx} & 0 \\ 0 & 0 & 0 & \Pi^{yy} \end{bmatrix}.$$

By defining $A = \Pi^{\eta}{}_{\eta}$, $B = \Pi^{x}{}_{x}$ and $C = \pi^{y}{}_{y}$, we can get

$$\begin{aligned}
A &= -\frac{4}{3}\frac{\eta\epsilon^{3/4}}{\tau} - \frac{\tau_\pi}{\epsilon^{1/4}}\left(\partial_\tau A + \frac{4}{3\tau}A\right) + \frac{\lambda_1}{3\epsilon}\left(2A^2 - B^2 - C^2\right) \\
B &= +\frac{2}{3}\frac{\eta\epsilon^{3/4}}{\tau} - \frac{\tau_\pi}{\epsilon^{1/4}}\left(\partial_\tau B + \frac{4}{3\tau}B\right) + \frac{\lambda_1}{3\epsilon}\left(2B^2 - A^2 - C^2\right) \\
C &= +\frac{2}{3}\frac{\eta\epsilon^{3/4}}{\tau} - \frac{\tau_\pi}{\epsilon^{1/4}}\left(\partial_\tau C + \frac{4}{3\tau}C\right) + \frac{\lambda_1}{3\epsilon}\left(2C^2 - B^2 - A^2\right),
\end{aligned}$$

and

$$\partial_\tau\epsilon + \frac{4}{3\tau}\epsilon + \frac{A}{\tau} = 0.$$

Special solutions

We assume $A = \frac{8}{3}\epsilon$, which gives

$$\epsilon = \frac{\mathcal{C}^4}{\tau^4}. \qquad \text{as compared to} \qquad \epsilon_{\text{Ideal}} \propto \frac{1}{\tau^{4/3}}$$

- Assume $A = -2B = -2C$, thus, it is very straightforward to find that

$$A = \frac{\epsilon}{\lambda_1}\left[\left(1 - \frac{8\tau_\pi}{3\mathcal{C}}\right) \pm \sqrt{\left(1 - \frac{8\tau_\pi}{3\mathcal{C}}\right)^2 + \frac{8}{3}\frac{\eta\lambda_1}{\mathcal{C}}}\right],$$

  which indicates

$$\mathcal{C} = \frac{3\eta - 16\tau_\pi}{8\lambda_1 - 6}.$$

  In fact, there are four sets of solutions for A, B, C as in total.

- Truly non-perturbative solution of the above non-linear equation.

$$\epsilon_{\text{2nd}} \propto \frac{1}{\tau^4} \qquad \text{as compared to} \qquad \epsilon_{\text{Ideal}} \propto \frac{1}{\tau^{4/3}}$$

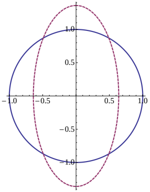
Elliptic flow solutions

Consider the static solution in $dS_3 \times \mathbb{R}$ (Gubser solution)

$$d\hat{s}^2 \equiv \frac{ds^2}{\tau^2} = \frac{-d\tau^2 + dx_\perp^2 + x_\perp^2 d\phi^2}{\tau^2} + d\eta^2$$
$$= -d\varrho^2 + \cosh^2\varrho(d\Theta^2 + \sin^2\Theta d\phi^2) + d\eta^2$$

$$\hat{u}_\varrho = -1, \qquad \hat{u}_\eta = \hat{u}_\Theta = \hat{u}_\phi = 0 \quad \Rightarrow$$
$$u_\tau = -\cosh\left[\tanh^{-1}\frac{2\tau x_\perp}{L^2+\tau^2+x_\perp^2}\right], \quad u_\perp = \sinh\left[\tanh^{-1}\frac{2\tau x_\perp}{L^2+\tau^2+x_\perp^2}\right]$$

Use Zhukovski transform to get approximate elliptic solution (small eccentricity)



$$\int_0^{2\pi} d\phi\,(u_1^2 - u_2^2) \approx \int d\phi \left(\cos 2\phi\, u_\perp^2 - \frac{2\sin 2\phi}{x_\perp} u_\phi u_\perp\right)$$
$$= 2\pi a^2 u_{\perp 0}\left(\delta u_\perp - \frac{\delta u_\phi}{x_\perp}\right) = \frac{16\pi a^2\tau^2 L^2}{(L^2+x_\perp^2)^3} > 0\,,$$

Elliptic flow

The Hitchhiker′s Guide to the Hydrodynamics

Bo-Wen Xiao

Introduction

Conformal Soliton Solution

Rotating Flow

Bjorken Flow at Second Order

Elliptic flow

Summary

Momentum space anisotropy $\tau^2 \ll L^2$ and $a^2 \ll L^2$

$$\left.\frac{\int dxdy(T_{xx}-T_{yy})}{\int dxdy(T_{xx}+T_{yy})}\right|_{\delta\mathcal{E}} = \frac{20a^2\tau^2}{3L^4}\left[-\frac{80}{77}+\frac{3\eta_0}{2C}\left(\frac{L}{2\tau}\right)^{2/3}-\frac{3264\eta_0^2}{385C^2}\left(\frac{L}{2\tau}\right)^{4/3}\right],$$

$$\left.\frac{\int dxdy(T_{xx}-T_{yy})}{\int dxdy(T_{xx}+T_{yy})}\right|_{\delta u} = \frac{20a^2\tau^2}{3L^4}\left[\frac{6}{7}-\frac{3\eta_0}{2C}\left(\frac{L}{2\tau}\right)^{2/3}+\frac{513\eta_0^2}{70C^2}\left(\frac{L}{2\tau}\right)^{4/3}\right].$$

Total ϵ_p

$$\epsilon_p(\tau) = \frac{20a^2\tau^2}{3L^4}\left[-\frac{2}{11}-\frac{177\eta_0^2}{154C^2}\left(\frac{L}{2\tau}\right)^{4/3}\right].$$

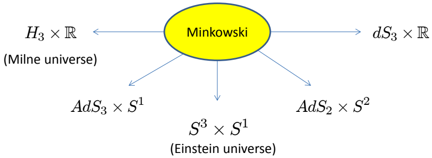
Comments:

- Negative $\epsilon_p(\tau)$ may be model dependent, due to nonzero initial radial flow. (Numerical check?)
- δu part is reflecting the genuine v_2 physics.
- Viscous correction agrees with empirical formula. [Bhalerao *et al.*, 05]

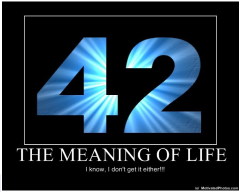
$$\left.\frac{\epsilon_p(\tau)}{\epsilon_p^{ideal}(\tau)}\right|_{\delta u} \sim \frac{1}{1+\frac{\eta_0}{C}\left(\frac{L}{\tau}\right)^{2/3}} \sim \frac{1}{1+\frac{L^2}{\sigma dN/dY}} \sim \frac{v_2}{v_2^{ideal}}.$$

Summary

Conformal Transform helps to find analytical solutions to the hydrodynamic equations. $ds^2 = \Lambda^2 \hat{g}_{\mu\nu} d\hat{x}^\mu d\hat{x}^\nu \equiv \Lambda^2 d\hat{s}^2$



[Hatta]



Bring in more analytical insights into hydrodynamics. (Give us reason behind "42")

- Conformal soliton solution
- Solutions with vortices.
- Non-perturbative Bjorken flow solutions.
- Analytical study of elliptic flow.